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MEASURE OF CURVES IN GRADED GROUPS

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ABSTRACT. We consider continuously differentiable curves embedded in graded groups, along with the points of these curves that have the maximal transversality with respect to the grading of the group. We prove a blow-up theorem at these points and we show that the remaining points are negligible with respect to the Hausdorff measure whose dimension equals the Hausdorff dimension of the curve. This leads us to an area-type formula for the intrinsic spherical Hausdorff measure of any of these curves embedded in an arbitrary graded group.

1. Introduction

In the seminal paper by M. Gromov [4], an interesting formula for the Hausdorff dimension of submanifolds in arbitrary Carnot–Carathéodory spaces can be found, see Section 0.6 B. Once we have the Hausdorff dimension, the subsequent question is to investigate whether the corresponding Hausdorff measure restricted to the submanifold is positive, locally finite and can be computed by an integral formula.

This is exactly the case, when a suitable “negligibility condition” is assumed to be satisfied by the single submanifold, according to the results of [7]. In fact, here an area-type formula for $\mathcal{S}^q \llcorner \Sigma$ is established, where q is the degree of the submanifold Σ contained in a graded group \mathbb{G} and \mathcal{S}^q represents the spherical Hausdorff measure with respect to the fixed homogeneous distance of the group. Notice that 0.6 B of [4] joined with Remark 4.2 of [9] shows that the degree of Σ coincides with its Hausdorff dimension. In short, the

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validity of an area type formula for $\mathcal{S}^q(\Sigma)$ is a consequence of the validity of the negligibility condition, that we recall in the following definition.

DEFINITION 1 (Negligibility condition). Let $\Sigma \subset \mathbb{G}$ be an embedded submanifold of degree q , where \mathbb{G} is a stratified group. We say that Σ satisfies the *negligibility condition* if we have

$$(1) \quad \mathcal{H}^q(\{x \in \Sigma : d_\Sigma(x) < q\}) = 0.$$

In the sequel, we always consider curves as one dimensional C^1 smooth embedded submanifolds. The notion of pointwise degree $d_\Sigma(\cdot)$ for curves is given in Definition 2.

For instance, in Heisenberg groups \mathbb{H}^n any horizontal vector field has degree one and the vertical vector field, belonging to the second layer, has degree two. Horizontal curves in Heisenberg groups are characterized by the condition of having pointwise degree equal to one everywhere. This remark clearly extends to all stratified groups. If the velocity vector of a curve in \mathbb{H}^n has at some point a nonvanishing component in the direction of degree two, then it has degree two at that point. The degree of a curve is defined as the maximum among all pointwise degrees, hence it takes somehow into account the global behaviour of the curve. Clearly, the negligibility condition becomes trivial for all smooth horizontal curves in \mathbb{H}^n and more generally for all horizontal submanifolds in stratified groups, since these manifolds have all degree equal to one.

Let us consider a curve Σ in \mathbb{H}^n that is the C^1 smooth union of some horizontal curves and some curves tangent to the vertical vector field. Clearly Σ has degree two and the fact that horizontal curves have finite one dimensional Hausdorff measure makes the negligibility condition obviously satisfied. In a general stratified group, one can consider smooth curves that are made by joining smooth pieces of different degrees and easily realize that the piece of highest degree determines the Hausdorff measure of the curve, making the remaining pieces negligible. The technical difficulty in the study of the negligibility condition is that in an arbitrary C^1 smooth curve, and more generally in a smooth submanifold, the closed subset $\{x \in \Sigma : d_\Sigma(x) \leq q - 1\}$ might be very far from having this structure, namely, it might be very little regular. We wish to point out that the validity of (1) for arbitrary submanifolds in arbitrary stratified groups is still an intriguing open question, where the regularity of the submanifold is an important parameter to be fixed and that depends on both the dimension of the submanifold and on the step of the group. On the other hand, condition (1) holds for C^1 smooth non-horizontal submanifolds, [6], [8], for $C^{1,1}$ smooth submanifolds of two step groups, [9], and for $C^{1,1}$ smooth submanifolds of the Engel group, [5]. The proofs in these different cases rely on different approaches.

The present work continues this program, following the recent approach of [5]. Precisely, we show that (1) also holds for all C^1 smooth curves of an arbitrary graded group, namely, we have the following theorem.

THEOREM 1.1 (Negligibility). *Every curve contained in a graded group satisfies the negligibility condition.*

As it is clear from [5], proving negligibility requires some ad hoc arguments to treat the blow-up of different types of “singular points”, that can occur in the study of the single submanifold. This singular behaviour is somehow given by the interplay between the degree of the submanifold $d(\Sigma)$ and the pointwise degree at the singular point $d_\Sigma(x)$. In particular, this implies a precise differential constraint on the local Taylor expansion of the submanifold near the singular point. In other terms, we show that all possible singularities of curves can be handled through a single argument by induction, that only requires C^1 regularity.

Notice that regularity is important, since the precision needed in the local description of the submanifold depends on its “local transversality”. Loosely speaking, in submanifolds of higher dimension, low regularity allows for a larger size of the set of points with low degree. This fact has been carefully studied in Heisenberg groups by Z. M. Balogh [1].

As mentioned above, the negligibility condition implies an integral formula for the spherical Hausdorff measure of the submanifold. This in turn follows by an intrinsic blow-up at each point of maximum degree, [7]. However, this blow-up requires $C^{1,1}$ smoothness, so that Theorem 1.1 would only lead us to an area-type formula for curves of class $C^{1,1}$. This stronger regularity somehow conflicts with the C^1 smoothness needed for the negligibility condition.

In fact, the second issue of this paper concerns the possibility to perform an intrinsic blow-up of curves at points of maximum degree, under C^1 regularity. The main point of the approach in [7] is that of finding a local reparametrization of the submanifold, using a family of curves on right neighbourhoods of the origin that solve a Lipschitz ODEs on the submanifold. For C^1 smooth submanifolds this approach fails, since the solutions of the same continuous ODEs are no longer unique. Clearly, in the case of curves, two of these solutions are enough to parametrize a neighbourhood of the point of maximum degree. This suggests to find a suitable blow-up reparametrization of the curve around its point of maximum degree. This yields the correct blow-up estimates, according to Proposition 3.1, and it allows us to establish the following theorem.

THEOREM 1.2. *Let \tilde{g} be any Riemannian metric on \mathbb{G} and let Σ be a C^1 smooth curve such that $d(\Sigma) = q$. If $x \in \Sigma$ and $d_\Sigma(x) = q$, then*

$$(2) \quad \lim_{r \rightarrow 0^+} \frac{\tilde{\mu}_1(\Sigma \cap B_{x,r})}{r^q} = \frac{\theta(\tau_\Sigma^q(x))}{|\tau_\Sigma^q(x)|},$$

where $\tilde{\mu}_1$ is the Riemannian measure on Σ with respect to the metric \tilde{g} .

Joining Theorem 1.1 with Theorem 1.2, by standard differentiability theorems on measures, see [3], we get the following corollary.

COROLLARY 1.1. *Let \tilde{g} be a fixed Riemannian metric on \mathbb{G} . If Σ is a C^1 curve such that $d(\Sigma) = q$, then*

$$(3) \quad \int_{\Sigma} \theta(\tau_{\Sigma}^q(x)) d\mathcal{S}^q(x) = \int_{\Sigma} |\tau_{\Sigma}^q(x)| d\tilde{\mu}_1(x).$$

The possible anisotropy of the homogeneous distance d makes the metric factor $\theta(\tau_{\Sigma}^q(x))$ a function that depends on the direction $\tau_{\Sigma}^q(x)$, see Definition 4. Recall that, according to Section 2.1, the stratified group \mathbb{G} can be decomposed into the direct sum of linear subspaces H_j of degree j , where $j = 1, \dots, \iota$. Using this notation, we say that the distance d has a *symmetry of degree q* if for all $\tau \in H_q$ we have

$$(4) \quad c_q = \theta(\tau) = \mathcal{H}_{|\cdot|}^1(\text{span}\{\tau\} \cap B_1),$$

then under this assumption the integral formula (3) becomes

$$(5) \quad c_q \mathcal{S}^q(\Sigma) = \int_{\Sigma} |\tau_{\Sigma}^d(x)| d\tilde{\mu}_1(x).$$

In connection with Gromov's dimension comparison problem studied in [2], we wish to regard our work from a different perspective. If \mathbb{G} is a stratified group of step ι , then for any integer $q = 1, \dots, \iota$, one easily observes that the subgroup L_q spanned by an element e_i of degree $d_i = q$ has Hausdorff dimension equal to q . A consequence of our results is that these integers are the only possible Hausdorff dimensions of C^1 smooth curves. In addition, each of these curves has an area-type formula (3) that computes their spherical Hausdorff measure.

2. Definitions and standard facts

A graded group \mathbb{G} with topological dimension n is a simply connected nilpotent Lie group with Lie algebra \mathcal{G} having the grading $\mathcal{G} = V_1 \oplus \dots \oplus V_{\iota}$, that satisfies the conditions $[V_i, V_j] \subset V_{i+j}$ for all $i, j \geq 1$, where $V_j = \{0\}$ whenever $j > \iota$. The integer ι is called the *step* of \mathbb{G} . Recall that the family of graded groups strictly contains the well-known family of stratified groups. The grading allows for defining dilations $\delta_r : \mathcal{G} \rightarrow \mathcal{G}$ as follows $\delta_r(\sum_{j=1}^{\iota} v_j) = \sum_{j=1}^{\iota} r^j v_j$, for all $r > 0$, where we have set $v = \sum_{j=1}^{\iota} v_j$ and $v_j \in V_j$. Note that simply connected nilpotent Lie groups are diffeomorphic to their Lie algebra through the exponential mapping $\exp : \mathcal{G} \rightarrow \mathbb{G}$, hence dilations are automatically defined as group isomorphisms of \mathbb{G} and will be denoted by the same symbol δ_r .

We say that ρ is a homogeneous distance on \mathbb{G} if it is a continuous distance of \mathbb{G} satisfying the following conditions:

$$d(zx, zy) = d(x, y) \quad \text{and} \quad d(\delta_r(x), \delta_r(y)) = r d(x, y) \quad \text{for all } x, y, z \in \mathbb{G}, r > 0.$$

We denote by \mathcal{H}^q and \mathcal{S}^q the q -dimensional Hausdorff and spherical Hausdorff measures induced by a fixed homogeneous distance ρ , respectively. Open balls with center at x and radius $r > 0$ with respect to d will be denoted by $B_{x,r}$ and the corresponding closed balls will be denoted by $D_{x,r}$.

2.1. Graded groups as vector spaces. The fact that the exponential mapping $\exp: \mathcal{G} \rightarrow \mathbb{G}$ is a diffeomorphism allows us to identify \mathbb{G} with its Lie algebra as follows. We set $H_j = \exp V_j$ for all $j = 1, \dots, \iota$ and equip \mathbb{G} with a structure of graded vector space satisfying $\mathbb{G} = H_1 \oplus \dots \oplus H_\iota$. This direct sum has the corresponding *canonical projections*

$$p_j: \mathbb{G} \rightarrow H_j, \quad p_j \left(\sum_{l=1}^{\iota} x_l \right) = x_j, \quad \text{where } x_l \in H_l \text{ for all } l = 1, \dots, \iota.$$

The group operation

$$(6) \quad x \cdot y = x + y + Q(x, y)$$

has a polynomial form, where Q is given by the Baker–Campbell–Hausdorff formula. We define the integers $m_0 = 0$ and $m_j = \sum_{l=1}^j \dim H_l$ for any $j = 1, \dots, \iota$. A basis (e_1, \dots, e_n) of \mathbb{G} is *graded* if

$$(e_{m_{j-1}+1}, \dots, e_{m_j}) \quad \text{is a basis of } H_j$$

for all $j = 1, \dots, \iota$. In the sequel, a graded basis will be understood whenever \mathbb{G} is identified with \mathbb{R}^n . Declaring this basis orthonormal, we have fixed both a scalar product on \mathbb{G} and a left invariant metric g on \mathbb{G} with respect to the group operation. With slight abuse of notation, we use the same notation to denote both the length $|v|$ of a vector $v \in T_y \mathbb{G}$ with respect to g and to denote the norm $|x|$ of an element x of \mathbb{G} . Notice that the metric g at the origin exactly coincides with the fixed scalar product of \mathbb{G} , since g is defined as the unique left invariant Riemannian metric on \mathbb{G} with this property. All the graded bases we consider are understood to be orthonormal with respect to the underlying scalar product g .

The basis (e_i) along with its coordinates (x_i) automatically inherit a *degree* from the layers H_j , namely,

$$d_j = k \quad \text{if and only if} \quad e_j \in H_k.$$

Elements of H_k have degree k and any coordinate x_j of an element $\sum_{j=1}^n x_j e_j \in \mathbb{G}$ has degree d_j . Taking into account this notion, dilations on \mathbb{G}

can be written as

$$(7) \quad \delta_r \left(\sum_{l=1}^n x_l e_l \right) = \sum_{l=1}^n r^{d_l} x_l e_l.$$

Notice that a change of system of graded coordinates is a linear isomorphism, hence the class of polynomials on \mathbb{G} is well defined. In fact, a polynomial on \mathbb{G} is simply a standard polynomial on \mathbb{R}^n when some fixed graded coordinates are fixed on \mathbb{G} . Let us stress that a linear isomorphism $I : \mathbb{R}^n \rightarrow \mathbb{R}^n$ arising from a change of graded coordinates automatically preserves the degrees of all coordinates, namely,

$$I(e_j) \in \text{span}\{e_l : l = m_{d_j-1} + 1, \dots, m_{d_j}\} \quad \text{for all } m_{d_j-1} + 1 \leq j \leq m_{d_j}.$$

For any $\alpha \in \mathbb{N}^n$, we define the monomial

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{where } d(\alpha) := \sum_{i=1}^n d_i \alpha_i$$

is the *homogeneous degree* of x^α . By (7), we have

$$(\delta_r x)^\alpha = r^{d(\alpha)} \prod_{j=1}^n x_j,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$. A polynomial is *d-homogeneous* if it is a linear combination of monomials of homogeneous degree equal to d . Thus, a polynomial P is *d-homogeneous* if and only if $P(\delta_r x) = r^d P(x)$ for all $x \in \mathbb{R}^n$ and $r > 0$. In this case, we also say that the *homogeneous degree* of P is d .

The degree transmits itself to the unique left invariant vector fields X_j of \mathbb{G} such that $X_j(0) = e_j$. It follows that X_j has degree d_j and has the polynomial form

$$(8) \quad X_j(x) = \partial_{x_j} + \sum_{l: d_l > d_j} a_j^l(x) \partial_{x_l}$$

with respect to the understood graded coordinates (x_j) . The polynomials a_j^l in the previous formula are $(d_l - d_j)$ -homogeneous. If (X_1, \dots, X_n) is a frame of left invariant vector fields generated by a graded basis (e_1, \dots, e_n) of \mathbb{G} , then we will say that this frame is also *graded*.

Next, we introduce the notion of degree of curves in a graded group. We address the reader to the original work, [7], for more details and the more general notion of degree of a submanifold.

DEFINITION 2 (Degree of curves). Let Σ be a curve of a graded group \mathbb{G} and let $x \in \Sigma$. Let $\tau \in T_x \Sigma \setminus \{0\}$ and let $\tau = \sum_{j=1}^n \lambda_j X_j(x)$. The *pointwise degree* of Σ at x is the integer $d_\Sigma(x) = \max\{d_j : \lambda_j \neq 0\}$. The *degree* of Σ is the integer $d(\Sigma) = \max_{x \in \Sigma} d_\Sigma(x)$.

The left translations of the group are denoted by $l_x : \mathbb{G} \rightarrow \mathbb{G}$, $l_x(y) = x \cdot y$.

DEFINITION 3. Let \tilde{g} be a Riemannian metric in \mathbb{G} , let Σ be a curve of \mathbb{G} and let $\gamma : (a, b) \rightarrow \mathbb{G}$ be a C^1 smooth local parametrization of Σ , with $t \in (a, b)$ and $x = \gamma(t)$. We define the *unit tangent vector* of Σ at x as $\tau_\Sigma(x) = \dot{\gamma}(t)/|\dot{\gamma}(t)|_{\tilde{g}} \in T_x\Sigma$. If $p_j : \mathbb{G} \rightarrow H_j$, $1 \leq j \leq \iota$ is the canonical projection of \mathbb{G} , and we have identified \mathbb{G} with $T_0\mathbb{G}$, then we define the *j-projection of $\tau_\Sigma(x)$* by

$$(9) \quad \tau_\Sigma^j(x) = dl_x(p_j(dl_{x^{-1}}(\tau_\Sigma(x)))) \in T_x\mathbb{G} \cap dl_x(H_j).$$

DEFINITION 4 (Metric factor). Let $x \in \mathbb{G}$, let $\tau \in T_x\mathbb{G}$ and set $\tau_0 = dl_{x^{-1}}\tau \in T_0\mathbb{G}$. The *metric factor* is defined by

$$(10) \quad \theta(\tau) = \mathcal{H}_{|\cdot|}^1(\text{span}\{\tau_0\} \cap B_1),$$

where $|\cdot|$ is the fixed scalar product on \mathbb{G} and B_1 is the open unit ball with respect to the fixed homogeneous distance d .

Clearly, the metric factor is constant on directions of a left invariant vector field.

2.2. Some density estimates of geometric measure theory. Our arguments are based on the following elementary fact, see for instance 2.10.19 of [3].

LEMMA 1. Let X be a metric space, let μ be a Borel measure on X , let $a > 0$ and let $\{V_i\}_{i \in \mathbb{N}}$ be an open covering of X such that $\mu(V_i) < \infty$. If $Z \subset X$ is any Borel set and $\limsup_{r \rightarrow 0^+} r^{-a}\mu(D_{x,r}) \geq \kappa > 0$ whenever $x \in Z$, then $\mu(Z) \geq \kappa \mathcal{S}^a(Z)$.

The symbol \mathcal{S}^a in the previous lemma denotes the a -dimensional spherical Hausdorff measure constructed by the size function $\zeta_a(D_{x,r}) = r^a$ and $D_{x,r}$ is the closed ball of center x and radius r . We have the following corollary.

COROLLARY 2.1. Let $a > 0$ be any positive number and let Σ be C^1 curve of a graded group \mathbb{G} . Let μ be the left invariant Riemannian measure of \mathbb{G} restricted to Σ and let Z be a Borel set of Σ such that $\limsup_{r \rightarrow 0^+} r^{-a}\mu(D_{z,r}) = +\infty$, whenever $z \in Z$. This implies that $\mathcal{S}^a(Z) = 0$.

We are going to apply this corollary when a is the degree of the curve Σ .

3. Blow-ups and negligibility

This section is devoted to the proofs of our main results.

PROPOSITION 3.1. Let $\gamma : (-1, 1) \rightarrow \mathbb{G}$ be C^1 smooth embedding, let Σ be its image and fix $d(\Sigma) = q$. If $\gamma(0) = 0$ and $d_\Sigma(0) = q$, then there exists a graded basis (e_1, \dots, e_n) such that for all $j = 1, 2, \dots, n$ we have

$$(11) \quad \gamma_i(t) = o(t^{d_i/q}) \quad \text{whenever } i \neq i_0 = m_{q-1} + 1.$$

Proof. The grading $\mathbb{G} = H_1 \oplus \cdots \oplus H_\ell$ has the associated canonical projections $p_j : \mathbb{G} \rightarrow H_j$, hence $p_q(\dot{\gamma}(0)) \neq 0$. We set $e_{i_0} = p_q(\dot{\gamma}(0))/|p_q(\dot{\gamma}(0))|$. We complete this vector to a graded basis (e_1, \dots, e_n) of \mathbb{G} and consider the corresponding frame of left invariant vector fields (X_1, \dots, X_n) . We have

$$\dot{\gamma}(t) = \sum_{s=1}^n C^s(\gamma(t)) X_s(\gamma(t)),$$

where the functions $t \rightarrow C^s(\gamma(t))$ are continuous, vanish on a neighbourhood of 0 whenever $d_s > q$, $C^{i_0}(0) \neq 0$ and for all s such that $i_0 < s \leq m_q$ we have $C^s(0) = 0$. Let us introduce the homeomorphism $\eta : \mathbb{R} \rightarrow \mathbb{R}$ defined as $\eta(t) = (|t|^q \operatorname{sgn}(t))/q$. We consider the reparametrized curve $\sigma(t) = \gamma(\eta(t))$, hence

$$\sigma'(t) = |t|^{q-1} \sum_{s:d_s \leq q} C^s(\sigma(t)) X_s(\sigma(t)).$$

Thus, clearly $\sigma'_i(t) = o(t^{d_i-1})$ for all i such that $d_i < q$. Due to (8), setting $a_s^i \equiv \delta_s^i$ when $d_i = d_s$ and $a_s^i \equiv 0$ when $d_i < d_s$, we can write

$$X_j(x) = \partial_{x_j} + \sum_{l=1}^n a_j^l(x) \partial_{x_l}.$$

We consider the cases $d_i \geq q$, where the previous formula gives

$$(12) \quad \sigma'_i(t) = |t|^{q-1} \sum_{s:d_s \leq q} C^s(\sigma(t)) a_s^i(\sigma(t)).$$

We first consider all i 's such that $d_i = q$, therefore

$$\begin{aligned} \sigma'_i(t) &= |t|^{q-1} \sum_{s:d_s \leq q} C^s(\sigma(t)) a_s^i(\sigma(t)) \\ &= |t|^{q-1} C^i(\sigma(t)) + |t|^{q-1} \sum_{s:d_s < q} C^s(\sigma(t)) a_s^i(\sigma(t)). \end{aligned}$$

The polynomials $a_s^i(x)$ are $(q - d_s)$ -homogeneous, hence they only depend on the components x_l with $d_l < q$, for which $\gamma_l(t) = o(t^{d_l})$. Thus, $a_s^i(\sigma(t)) = o(t^{q-d_s})$ and

$$\sigma'_i(t) = |t|^{q-1} C^i(\sigma(t)) + o(t^{q-1}).$$

In the case $i \neq i_0$ and $d_i = q$, we have $C^i(0) = 0$, therefore $\sigma'_i(t) = o(t^{q-1})$. This is not true for $i = i_0$, since $C^{i_0}(0) \neq 0$.

Now, we consider all i 's such that $d_i = q+1$. We split (12) into two addends, getting

$$\sigma'_i(t) = |t|^{q-1} \sum_{s:d_s=q} C^s(\sigma(t)) a_s^i(\sigma(t)) + |t|^{q-1} \sum_{s:d_s < q} C^s(\sigma(t)) a_s^i(\sigma(t)).$$

In the second addend, the homogeneous degree of a_s^i is less than or equal to q , hence we can only conclude that $a_s^i(\gamma(t)) = O(t^{q+1-d_s}) = o(t)$. It follows that

$$\sigma'_i(t) = |t|^{q-1} \sum_{s:d_s=q} C^s(\sigma(t)) a_s^i(\sigma(t)) + o(t^q).$$

Furthermore, $C^s(\sigma(t)) = o(1)$ whenever $s \neq i_0$ and $d_s = q$ and $a_s^i(\sigma(t)) = O(t)$. We have then established

$$(13) \quad \sigma'_i(t) = |t|^{q-1} C^{i_0}(\sigma(t)) a_{i_0}^i(\sigma(t)) + o(t^q).$$

Now, by definition of left invariant vector fields, and representing $Q(x, y)$ of (6) as $\sum_{i=1}^n Q_i(x, y) e_i$, we have $a_s^i(x) = \partial_{y_s} Q_i(x, 0)$ whenever $d_i > d_s$. Clearly $\text{span}\{X_{i_0}\}$ is a subalgebra of \mathcal{G} , then we can apply Lemma 2.5 of [7], that gives

$$Q_i(x, y) = \sum_{l \neq i_0, d_l < d_i} x_l R_l^i(x, y) + y_l S_l^i(x, y)$$

whenever $i \neq i_0$. In particular, we are lead to

$$(14) \quad a_{i_0}^i(x) = \sum_{l \neq i_0, d_l < d_i} x_l \partial_{i_0} R_l^i(x, 0).$$

In the previous steps, we have proved that $\sigma_l(t) = o(t^{d_l})$ for each l such that $d_l \leq q$ and $l \neq i_0$, hence the previous formula yields

$$a_{i_0}^i(\sigma(t)) = \sum_{l \neq i_0, d_l < d_i} \sigma_l(t) \partial_{i_0} R_l^i(\sigma(t), 0) = o(t^{d_i-d_{i_0}}),$$

where we have taken into account that $\partial_{i_0} R_l^i(\sigma(t), 0) = O(t^{d_i-d_0-d_l})$ for $d_i - d_{i_0} - d_l \geq 0$ and $\partial_{i_0} R_l^i(\sigma(t), 0) \equiv 0$ otherwise. In view of (13), we have proved that $\sigma'_i(t) = o(t^q)$.

To complete the proof, we argue by induction, assuming that $\sigma'_i(t) = o(t^{d_i-1})$ whenever i satisfies $1+q \leq d_i \leq k-1$ and $k \geq 2$. We consider the cases $d_i = k$. The next steps essentially repeat the previous argument. In fact, we now consider

$$\sigma'_i(t) = |t|^{q-1} \sum_{s:d_s=q} C^s(\sigma(t)) a_s^i(\sigma(t)) + |t|^{q-1} \sum_{s:d_s < q} C^s(\sigma(t)) a_s^i(\sigma(t)).$$

The homogeneous degree of a_s^i is less than or equal to $k-1$, hence $a_s^i(\gamma(t)) = O(t^{d_i-d_s})$. It follows that

$$\sigma'_i(t) = |t|^{q-1} \sum_{s:d_s=q} C^s(\sigma(t)) a_s^i(\sigma(t)) + o(t^{d_i-1}).$$

Since $C^s(0) = 0$ for $d_s = q$ and $s \neq i_0$, we have

$$C^s(\sigma(t)) a_s^i(\sigma(t)) = o(1) O(t^{d_i-d_s}) = o(t^{d_i-d_s}).$$

It follows that $\sigma'_i(t) = |t|^{q-1} C^{i_0}(\sigma(t)) a_{i_0}^i(\sigma(t)) + o(t^{d_i-1})$. Using formula (14) and the induction hypothesis, we also get $a_{i_0}^i(\sigma(t)) = o(t^{d_i-d_{i_0}})$, leading us to $\sigma'_i(t) = o(t^{d_i-1})$, that concludes the proof. \square

The previous proposition is the basic ingredient to establish the blow-up at points of maximum degree.

Proof of Theorem 1.2. For any sufficiently small $r > 0$, we have

$$(15) \quad \frac{\tilde{\mu}_1(\Sigma \cap B_{x,r})}{r^q} = \frac{1}{r^q} \int_{\Gamma^{-1}(B_{x,r})} |\dot{\Gamma}(t)|_{\tilde{g}} dt \\ = \int_{A_{x,r}} |\dot{\Gamma}(r^q t)|_{\tilde{g}} dt,$$

where $\Gamma: (-1, 1) \rightarrow \mathbb{G}$ is a local chart of Σ , $\Gamma(0) = x$ and $\Sigma \cap B_{x,r} = \Gamma(\Gamma^{-1}(B_{x,r}))$. We have also set $A_{x,r} = \{t \in (-1, 1) : \Gamma(\text{tr}^q) \in B_{x,r}\}$. Let us consider the curve $\gamma(t) = x^{-1} \cdot \Gamma(t)$ and observe that it parametrizes the translated submanifold $x^{-1}\Sigma$, that has same degree q , and $\gamma(0) = 0$. We have

$$A_{x,r} = \left\{ t \in (-r^{-q}, r^{-q}) : \sum_{i=1}^n \frac{\gamma_i(\text{tr}^q)}{r^{d_i}} e_i \in B_1 \right\}$$

and Proposition 3.1 yields a graded basis (e_1, \dots, e_n) with respect to which we have $\gamma_i(t) = o(|t|^{d_i/q})$, whenever $i \neq i_0$ and $i_0 = m_{q-1} + 1$. For these integers

$$\frac{\gamma_i(\text{tr}^q)}{r^{d_i}} = \frac{o(|\text{tr}^q|^{d_i/q})}{r^q} \rightarrow 0 \quad \text{as } r \rightarrow 0^+$$

and clearly $\gamma_{i_0}(\text{tr}^q) r^{-q} \rightarrow \gamma'_{i_0}(0)t$, where these limits are uniform with respect to t that varies on compact sets of \mathbb{R} . In particular, we have the L^1_{loc} convergence of the characteristic functions

$$\mathbf{1}_{A_{x,r}} \rightarrow \mathbf{1}_{\frac{1}{\gamma'_{i_0}(0)} S_0}, \quad \text{where } S_0 = \{t \in \mathbb{R} : t e_{i_0} \in B_1\}$$

and $\mathbf{1}_{A_{x,r}} \leq \mathbf{1}_{[-M_0, M_0]}$ for some $M_0 > 0$ and any $r > 0$ small. By Lebesgue's theorem,

$$(16) \quad \lim_{r \rightarrow 0^+} \frac{\tilde{\mu}_1(\Sigma \cap B_{x,r})}{r^q} = |\dot{\Gamma}(0)| \mathcal{L}^1\left(\frac{1}{\dot{\gamma}_{i_0}(0)} S_0\right) = \frac{|\dot{\Gamma}(0)|}{|\dot{\gamma}_{i_0}(0)|} \mathcal{L}^1(S_0).$$

We first observe that $|e_{i_0}| = 1$ gives $\mathcal{L}^1(S_0) = \mathcal{H}^1_{|\cdot|}(B_1 \cap \text{span}\{e_{i_0}\}) = \theta(\tau_\Sigma^q(x))$. We also observe that

$$\frac{\dot{\gamma}_{i_0} e_{i_0}}{|\dot{\Gamma}(0)|_{\tilde{g}}} = \frac{p_q(dl_{x^{-1}} \dot{\Gamma}(0))}{|\dot{\Gamma}(0)|_{\tilde{g}}} \\ = dl_{x^{-1}} \left(\frac{dl_x(p_q(dl_{x^{-1}} \dot{\Gamma}(0)))}{|\dot{\Gamma}(0)|_{\tilde{g}}} \right) = dl_{x^{-1}}(\tau_\Sigma^q(x)),$$

hence we have

$$|\tau_{\Sigma}^q(x)| = |dl_{x^{-1}}(\tau_{\Sigma}^q(x))| = \left| \frac{\dot{\gamma}_{i_0} e_{i_0}}{|\dot{\Gamma}(0)|_{\bar{g}}} \right| = \frac{|\dot{\gamma}_{i_0}|}{|\dot{\Gamma}(0)|_{\bar{g}}}.$$

The last equalities joined with (16) lead us to (2), concluding the proof. \square

PROPOSITION 3.2. *Let $\gamma : (-1, 1) \rightarrow \mathbb{G}$ be C^1 parametrization of a curve Σ , that is the image of γ . We also assume that $\gamma(0) = 0$ and $d_{\Sigma}(0) < d(\Sigma)$. If (e_1, \dots, e_n) is any graded basis with $\gamma = \sum_{j=1}^n \gamma_j e_j$, then we have a constant $\sigma > 0$ along with nonnegative functions $\varepsilon_j : (0, \sigma) \rightarrow \mathbb{R}$ such that for all $t \in (-\sigma, \sigma)$ and $j = 1, 2, \dots, n$, we have*

$$(17) \quad |\gamma_j(t)| \leq \varepsilon_j(|t|)|t|^{p_j},$$

where $p_j = d_j/d(\Sigma)$ and $\varepsilon_j(r) \rightarrow 0^+$ as $r \rightarrow 0^+$.

Proof. There exists a neighbourhood $I \subset \mathbb{R}$ of 0 such that

$$(18) \quad \dot{\gamma}(t) = \sum_{j=1}^n \lambda_j(t) X_j(\gamma(t)) = \sum_{j=1}^n \dot{\gamma}_j(t) \partial_{x_j}.$$

We observe that the second equality and the C^1 smoothness of γ imply that all λ_j are continuous. Let us first consider the case $d_j \leq d_{\Sigma}(0)$. Here we have $p_j < 1$ and C^1 continuity of γ_j yields $|\dot{\gamma}_j(t)| < \max\{2|\dot{\gamma}_j(0)|, 1\}$ for $|t|$ small enough. Therefore, as $\gamma_j(0) = 0$, we have $|\gamma_j(t)| < \max\{2|\dot{\gamma}_j(0)|, 1\}|t|$. This proves the validity of (17), where one sets $\varepsilon_j(r) = \max\{2|\dot{\gamma}_j(0)|, 1\}r^{1-p_j}$. If $d_{\Sigma}(0) < d_j \leq d(\Sigma)$, then we must have $\dot{\gamma}_j(0) = 0$. Thus, introducing the nondecreasing function $f_j(r) := \max_{|s| \leq r} |\dot{\gamma}_j(s)|$ we have

$$\lim_{r \rightarrow 0^+} f_j(r) = 0 \quad \text{and} \quad |\dot{\gamma}_j(t)| \leq f_j(t).$$

Therefore $|\gamma_j(t)| \leq f_j(|t|)|t| = \varepsilon_j(|t|)|t|^{p_j}$, where we have set $\varepsilon_j(r) = f_j(r)r^{1-p_j}$. Now, to prove the validity of (17) for all $j = 1, \dots, n$, we use the following induction argument. Suppose that for every n -dimensional stratified group \mathbb{G} and every Σ and γ satisfying our assumptions, the claim (17) holds for all $j = 1, 2, \dots, k-1$, where $k \leq n$. We have to show that this yields (17) for $j = k$.

If $k \in \mathbb{N}$ is such that $d_k \leq d(\Sigma)$, then (17) holds, due to the previous arguments. Let us consider the case $d_k > d(\Sigma)$. Joining (8) with (18), we have

$$\dot{\gamma}_k(t) = \lambda_k(t) + \sum_{d_j < d_k} \lambda_j(t) a_j^k(\gamma(t)).$$

By definition of degree, if $d_j > d(\Sigma)$, then λ_j everywhere vanishes. Therefore it remains to estimate $a_j^k \circ \gamma$ whenever $d_j \leq d(\Sigma)$. It follows that

$$\dot{\gamma}_k(t) = \sum_{j: d_j \leq d(\Sigma)} \lambda_j(t) a_j^k(\gamma(t)).$$

Recall that a_j^k is a polynomial $(d_k - d_j)$ -homogeneous. Thus, it only depends on variables x_l 's with $d_l < d_k$. Since $l \rightarrow d_l$ is nondecreasing, a_j^k only depends on the variables x_l with $l < k$. It follows that a_j^k can be written as

$$\sum_{d(\alpha)=d_k-d_j} c_\alpha x^\alpha = \sum_{d(\alpha)=d_k-d_j} c_\alpha x_1^{\alpha_1} \cdots x_{k-1}^{\alpha_{k-1}}.$$

Taking into account the induction hypothesis, we get

$$\left| \sum_{d(\alpha)=d_k-d_j} c_\alpha \gamma(t)^\alpha \right| \leq \left(\sum_{d(\alpha)=d_k-d_j} c_\alpha \prod_{l=1}^{k-1} \varepsilon_j(t)^{\alpha_l} \right) |t|^{\sum_{l=1}^{k-1} p_l \alpha_l}.$$

We also observe that

$$\sum_{l=1}^{k-1} p_l \alpha_l = \sum_{l=1}^{k-1} \frac{d_l}{d(\Sigma)} \alpha_l = \frac{d_k - d_j}{d(\Sigma)} \geq p_k - 1.$$

It follows that

$$\left| \sum_{d(\alpha)=d_k-d_j} c_\alpha \gamma(t)^\alpha \right| \leq \left(\sum_{d(\alpha)=d_k-d_j} c_\alpha \prod_{l=1}^{k-1} \varepsilon_j(|t|)^{\alpha_l} \right) |t|^{p_k-1}.$$

Thus, we can find a nonnegative and nondecreasing function $\varepsilon_k(r)$ for $r \geq 0$ sufficiently small, only depending on all polynomials a_j^k and all ε_j for $j \leq k-1$ such that

$$|\dot{\gamma}_k(t)| \leq \varepsilon_k(|t|) |t|^{p_k-1} \quad \text{and} \quad \lim_{r \rightarrow 0^+} \varepsilon_k(r) = 0.$$

This immediately leads us to the validity of (17) for $j = k$ and completes the proof. \square

PROPOSITION 3.3. *Let Σ be a C^1 smooth curve in a graded group \mathbb{G} . Let μ be the Riemannian measure induced on Σ by the fixed left invariant metric g . We have that for every $x \in \Sigma$ such that $d_\Sigma(x) < d(\Sigma)$, there holds*

$$(19) \quad \lim_{r \rightarrow 0^+} \frac{\mu(\Sigma \cap D_{x,r})}{r^{d(\Sigma)}} = +\infty.$$

Proof. For each $j = 1, \dots, n$, we define the orthogonal projection $\mathbf{p}_j : \mathbb{G} \rightarrow \text{span}\{e_j\}$ associated to the graded basis (e_1, \dots, e_n) of \mathbb{G} . We define the box

$$\text{Box}_r = \{x \in \mathbb{G} : |\mathbf{p}_j(x)| \leq r^{d_j}\}.$$

Since $\delta_r(\text{Box}_1) = \text{Box}_r$ for all $r > 0$ and $\text{Box}_\lambda \subset D_1 \subset \text{Box}_{\lambda-1}$ for a fixed $\lambda > 0$, we get

$$(20) \quad \text{Box}_{r\lambda} \subset D_r \subset \text{Box}_{r/\lambda}.$$

Let $\gamma : (-1, 1) \rightarrow \mathbb{G}$ be a C^1 smooth embedding such that $\gamma(0) = 0$ and whose image is contained in the translated submanifold $x^{-1} \cdot \Sigma$. Let $q = d(\Sigma)$ and consider

$$\frac{\mu(\Sigma \cap D_{x,r})}{r^q} = \frac{\mu((x^{-1} \cdot \Sigma) \cap D_r)}{r^q} \geq \frac{\mu((x^{-1} \cdot \Sigma) \cap \text{Box}_{\lambda r})}{r^q}.$$

It follows that for $r > 0$ sufficiently small, we have

$$(21) \quad \frac{\mu(\Sigma \cap D_{x,r})}{r^q} \geq \frac{1}{r^q} \int_{I_{x,r}} |\dot{\gamma}(t)| dt.$$

In the previous inequality we have set $I_{x,r} = \gamma^{-1}(\text{Box}_{\lambda r}) = \bigcap_{j=1}^n A_j(x, r)$, where

$$A_j(x, r) = \{t \in (-1, 1) : |\gamma_j(t)| \leq (\lambda r)^{d_j}\}.$$

Let $0 < h < 1$ be arbitrarily fixed and let $0 < \alpha_j < \min\{1, \sigma\}$ be such that $\varepsilon_j(|t|) < h$ whenever $|t| \leq \alpha_j$, where ε_j and σ are as in Proposition 3.2. This proposition gives

$$\{t \in (-\alpha_j, \alpha_j) : h|t|^{d_j/q} \leq (\lambda r)^{d_j}\} \subset A_j(x, r)$$

for all $j = 1, \dots, n$. Now, we define the numbers

$$0 < \sigma_0 < \min\{\alpha_j : j = 1, \dots, n\} \quad \text{and} \quad \sigma_1 = \max\{\alpha_j : j = 1, \dots, n\},$$

where σ_0 is chosen such that $|\dot{\gamma}(t)| > |\dot{\gamma}(0)|/2$ for any $t \in (-\sigma_0, \sigma_0)$. Thus, we introduce the set

$$S_r = \{t \in (-\sigma_0, \sigma_0) : |t| \leq (\lambda^q r^q)/h^{q/\sigma_1}\}.$$

We observe that $S_r \subset A_j(x, r)$ for every $j = 1, \dots, n$ and for $r > 0$ sufficiently small

$$\frac{\mu(\Sigma \cap D_{x,r})}{r^q} \geq \frac{|\dot{\gamma}(0)|}{2r^q} |S_r| = \frac{|\dot{\gamma}(0)|}{2} \frac{\lambda^q}{h^{q/\sigma_1}}.$$

Taking into account the arbitrary choice of h , we have reached our claim. \square

Proof of Theorem 1.1. By Proposition 3.3, for each point $x \in \{y \in \Sigma : d_\Sigma(y) < d(\Sigma)\}$ the limit (19) holds. As a consequence, applying Corollary 2.1 with $a = d(\Sigma)$ our claim follows. \square

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